MA 3046 - Matrix Analysis

Problem Set 1 - Review of Prerequisite Materials

1. Solve the following system by Gaussian Elimination (without row interchanges) and back substitution:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 8 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & 1 \\ 3x_1 & - & 7x_2 & + & 4x_3 & = & 10 \end{array}$$

2. Solve the following system by Gaussian Elimination (with row interchanges to avoid zero pivots only) and back substitution:

$$\begin{array}{rcrcrcr} 3x_1 & - & x_2 & + & 2x_3 & = & 1 \\ 6x_1 & - & 2x_2 & + & 5x_3 & = & -1 \\ -3x_1 & + & 2x_2 & - & x_3 & = & -5 \end{array}$$

3. Consider the following matrix (unaugmented):

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 & 3 & 1 \\ 2 & 1 & -2 & 5 & 3 \\ -2 & 1 & 6 & -6 & 1 \end{bmatrix}$$

a. Using elementary row operations (with row interchanges only when necessary to remove zeros in pivot positions), reduce the appropriate matrix to echelon form, and determine for what right-hand sides ($\mathbf{b}$) the system:

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}$$

will be solvable.

b. What is the general solution for the homogeneous problem associated with this system?

c. What is the rank of $\mathbf{A}$?

d. What is the Row Reduced Echelon Form of $\mathbf{A}$?

e. Express each non-pivot (free) column in the row reduced echelon form as a linear combination of the pivot columns.

4. Consider the following matrix (unaugmented):

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 & 4 & 4 \\ -3 & -6 & 3 & -5 & -8 \\ 2 & 4 & -2 & 5 & 2 \end{bmatrix}$$

a. Using elementary row operations (with row interchanges only when necessary to remove zeros in pivot positions), reduce the appropriate matrix to echelon form, and determine for what right-hand sides ($\mathbf{b}$) the system:

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}$$

will be solvable.

b. What is the general solution for the homogeneous problem associated with this system?

c. What is the rank of $\mathbf{A}$?

5. Find the general solution to the following linear system:

$$\begin{array}{rcrcrcrcrcl} x_1 & - & x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & = & 3 \\ -3x_1 & + & 3x_2 & - & 3x_3 & - & 3x_4 & - & 12x_5 & = & 0 \\ 2x_1 & - & 2x_2 & + & 3x_3 & + & 2x_4 & + & 6x_5 & = & 3 \end{array}$$

What is the rank of the matrix associated with this system?

6. Consider the block matrices:

$$\mathbf{A} = \left[\begin{array}{c|cc} 4 & 0 & 0 \\ \hline -2 & 3 & -4 \\ 1 & 2 & 3 \end{array}\right], \quad \mathbf{B} = \left[\begin{array}{cc|c} -1 & -2 & -5 \\ \hline 5 & 4 & -4 \\ -6 & -7 & -4 \end{array}\right], \quad \mathbf{C} = \left[\begin{array}{c|cc} 2 & 1 & 1 \\ \hline 0 & 2 & 0 \\ -2 & -3 & 2 \end{array}\right]$$

Compute, if possible the following as both block matrix operations and "ordinary" operations and confirm that the same results occur. If a computation is not possible, explain why.

a. $\mathbf{A}\,\mathbf{B}$

b. $\mathbf{B}\,\mathbf{A}$

c. $\mathbf{A}\,\mathbf{C}$

d. $\mathbf{A} + \mathbf{B}$

e. $\mathbf{A} + \mathbf{C}$

7. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ -2 & 1 & 0 \end{bmatrix}$$

a. Find a sequence of elementary matrices which, when successively multiplying $\mathbf{A}$ on the left, reduce $\mathbf{A}$ to row-reduced echelon form (or, equivalently, which produce the same result as Gauss-Jordan elimination.)

b. Show that $\mathbf{A}^{-1}$ is precisely a product of the elementary matrices determined above.

8. Solve the following system by $\mathbf{L}\,\mathbf{U}$ Decomposition (without row interchanges) and forward/backward substitution:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 8 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & 1 \\ 3x_1 & - & 7x_2 & + & 4x_3 & = & 10 \end{array}$$

9. Determine which of the following are spanning sets for $\Re^3$. For those that are not, determine geometrically the dimension of the subspace of $\Re^3$ which they actually span.

a. $\{\, [1 \;\; 1 \;\; 1] \,\}$
b. $\{\, [\,1\,0\,0\,]\,,\ [\,0\,0\,1\,] \,\}$
c. $\{\, [\,1\,0\,0\,]\,,\ [\,0\,1\,0\,]\,,\ [\,0\,0\,1\,]\,,\ [\,1\,1\,1\,] \,\}$
d. $\{\, [\,1 \;\; 2 \;\; 1\,]\,,\ [\,2 \;\; 0 \;\; -1\,]\,,\ [\,4 \;\; 4 \;\; 1\,] \,\}$
e. $\{\, [\,1 \;\; 2 \;\; 1\,]\,,\ [\,2 \;\; 0 \;\; -1\,]\,,\ [\,4 \;\; 4 \;\; 0\,] \,\}$

10. a. Using the inner product, find the length of each of the following vectors:

(1.) $\mathbf{u} = [1 \;\; 2 \;\; 2]^T$
(2.) $\mathbf{v} = [\,1 \;\; 1 \;\; 3 \;\; -5]^T$
(3.) $\mathbf{w} = [\,1 \;\; 1 \;\; -2 \;\; 1 \;\; -1\,]^T$

b. Find the angle between each of the following sets of vectors.

(1.) $\mathbf{u} = [\,1 \;\; 2 \;\; 1\,]^T$ and $\mathbf{v} = [\,2 \;\; 0 \;\; -1\,]^T$
(2.) $\mathbf{u} = [\,1 \;\; 2 \;\; 1 \;\; 2 \;\; 1\,]^T$ and $\mathbf{w} = [\,2 \;\; 2 \;\; 2 \;\; 1 \;\; 1\,]^T$

11. Determine which of the following sets are orthogonal, orthonormal, or neither. Convert any that are orthogonal, but not orthonormal, to an equivalent orthonormal set.

a. $[\,1\ \ 1\ \ 1\,]^T\,,\ [\,2\ \ -1\ \ -1\,]\,,\ [\,1\ -2\ \ 1]^T$

b. $$\left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} , \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} , \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} , \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\}$$

c. $\{\,[\,1\ \ 1\ \ 1\ \ 1\,]^T\,,\ [\,1\ \ \ 1\ \ \ 3\ \ -5\,]^T\,\}$

12. Find the dimension of and a basis for each of the fundamental subspaces associated with the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 & 3 & 1 \\ 2 & 1 & -2 & 5 & 3 \\ -2 & 1 & 6 & -6 & 1 \end{bmatrix}$$

(Note this is the same matrix as in problem 3.)

13. Find the dimension of and a basis for each of the fundamental subspaces associated with the following matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 & 4 & 4 \\ -3 & -6 & 3 & -5 & -8 \\ 2 & 4 & -2 & 5 & 2 \end{bmatrix}$$

(Note this is the same matrix as in problem 4.)

14. Find the coordinates of

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

in terms of the ordered basis

$$\mathbf{B} = \left\{ \mathbf{b}^{(1)}\ ,\ \mathbf{b}^{(2)} \right\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$$

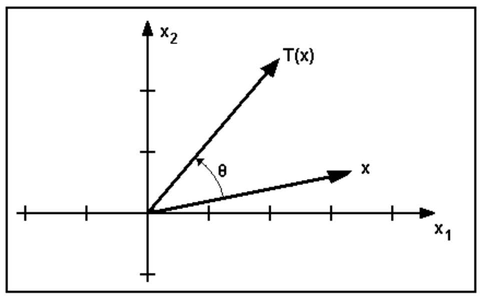
and confirm your answer geometrically.

15. Consider the transformation specified by

$$\mathbf{T}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - x_3 + x_4 \\ 2x_1 - 3x_2 + 3x_4 \\ x_3 - x_1 \end{bmatrix}$$

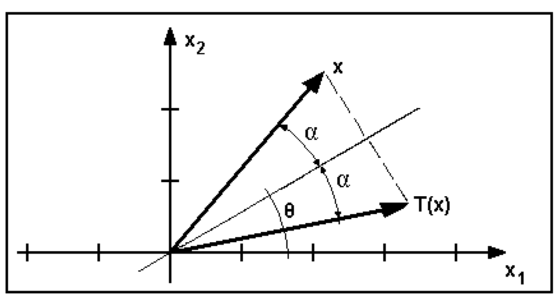
Find the coordinate matrix of $\mathbf{T}$ relative to the standard bases for its "input" and "output" spaces.

16. Consider the linear transformation $\mathbf{T}(\ )$ from $\Re^2$ to $\Re^2$ that simply rotates each "input" vector through a fixed, specified angle (θ), i.e.



Find the coordinate matrix for this transformation relative to the standard bases for its "input" and "output" spaces.

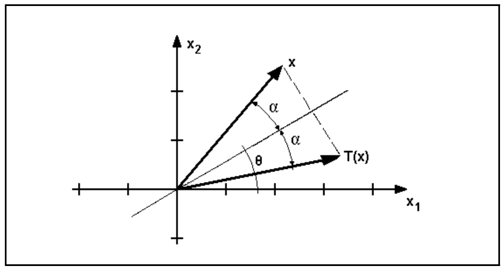
17. Consider the transformation that reflects any given "input" vector about a specified line at an angle of θ with respect to the x_1 axis, i.e.:



Show that the coordinate matrix for this transformation relative to the standard bases for its "input" and "output" spaces is:

$$[\mathbf{T}] = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

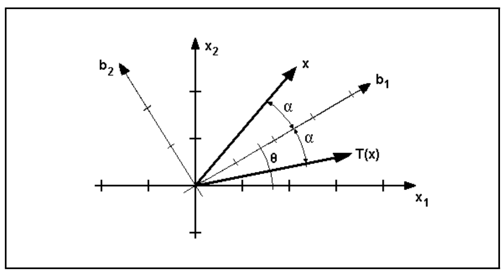
18. Consider the transformation that reflects any given "input" vector about a specified line at an angle of θ with respect to the x_1 axis, i.e.:



We have previously shown that, in terms of the standard (i.e. x_1, x_2) coordinate system, the matrix for this transformation is:

$$[\mathbf{T}] = \begin{bmatrix} (c^2 - s^2) & 2cs \\ 2cs & (s^2 - c^2) \end{bmatrix}$$

where $c = \cos(\theta)$ and $s = \sin(\theta)$. Consider now the non-standard coordinate system oriented at an angle of θ relative to the standard one, i.e.



Using the standard formula for similarity transformations, i.e.

$$[\mathbf{T}]_{\mathbf{B}} = \mathbf{B}^{-1}\,[\mathbf{T}]\,\mathbf{B}$$

find the matrix for this transformation relative to the above non-standard basis, and interpret your result geometrically.

19. For each of the following matrices:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 3 & -2 & 5 \\ 0 & 1 & 4 \\ 0 & -1 & 5 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0 & 6 & 3 \\ -1 & 5 & 1 \\ -1 & 2 & 4 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(i) Find all of the eigenvalues and their associated multiplicities.

(ii) For each different eigenvalue, find the dimension of and a basis for the associated eigenspace.

(iii) Determine whether the matrix is diagonalizable.

20. For each of the following symmetric matrices:

$$\mathbf{A} = \begin{bmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 2 & -2 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & 5 \\ -5 & 5 & 7 \end{bmatrix}$$

(i) Find all of the eigenvalues and their associated multiplicities.

(ii) For each different eigenvalue, find the dimension of and an orthonormal basis for the associated eigenspace.

(iii) Determine whether the matrix is diagonalizable.

21. Consider the matrices:

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 5 \\ 0 & -1 & 3 \\ -2 & 3 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(i) Find $\mathbf{S}^{-1}$.
(ii) Find $\mathbf{S}^{-1}\mathbf{A}\,\mathbf{S}$
(iii) Show that $\mathbf{A}$ and $\mathbf{S}^{-1}\mathbf{A}\,\mathbf{S}$ have the same characteristic polynomial (and hence the same eigenvalues).
(iv) Find the eigenvalues and eigenvectors of $\mathbf{S}^{-1}\mathbf{A}\,\mathbf{S}$.
(v) Show by direct computation that, if $\mathbf{v}^{(i)}$ is an eigenvector of $\mathbf{S}^{-1}\mathbf{A}\,\mathbf{S}$ corrensponding to λ_i, then $\mathbf{w}^{(i)} = \mathbf{S}\mathbf{v}^{(i)}$ is an eigenvector of $\mathbf{A}$ corresponding to the same eigenvalue.